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by

E. P. Aksenov
E. A. Grebenikov
V. G. Demin

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INTRODUCTION

This paper considers the polar orbits of Earth's artificial satellites. The potential of terrestrial attraction is approximated by the potential of two attraction centers situated at a certain imaginary distance. A qualitative analysis of the form of motion is conducted. The most interesting case for applications is studied in detail.

* * *

The problem of motion of an artificial satellite in a normal Earth's attraction field was considered in reference [1]. In a geocentrical rectangular system of coordinates with a fixed direction of axes, whose z-axis coincides with the Earth's rotation axis, the normal field potential is written as follows:

$$U = \frac{fM}{r}\left\{1 + \sum_{k=1}^{\infty}(-1)^k \left(\frac{c}{r}\right)^{2k} P_{2k}\left(\frac{z}{r}\right)\right\}, \qquad (1)$$

*O polyarnykh orbitakh iskusstvennykh sputnikov Zemli.

where f is the gravitation constant, M is the mass of the Earth, c is a certain constant, approximately equal to 210 km, $p_s\left(\frac{z}{r}\right)$ is a Legendre polynomial of the s - order, with $r = \sqrt{x^2 + y^2 + z^2}$.

Formula (1) may be transformed

$$U = \frac{fM}{2}\left(\frac{1}{\sqrt{x^2 + y^2 + (z - ci)^2}} + \frac{1}{\sqrt{x^2 + y^2 + (z + ci)^2}}\right),$$

where i is an imaginary unity.

It was shown in that paper that the problem considered is entirely resolved in quadratures. If we pass from the coordinates x, y, z and time t to new coordinates λ, μ, w and to a new independent variable τ by formulae

$$x = c\sqrt{(1 + \lambda^2)(1 - \mu^2)}\sin w, \qquad y = c\sqrt{(1 + \lambda^2)(1 - \mu^2)}\cos w, \tag{2}$$

$$z = -c\lambda\mu,$$

$$dt = (\lambda^2 + \mu^2)\,d\tau, \tag{3}$$

the general solution of this problem may be written as follows:

$$\int_{\mu_0}^{\mu} \frac{d\mu}{\sqrt{2h\mu^4 + 2(c_2 - h)\mu^2 - (2c_2 + c_1^2)}} = \tau + c_3, \tag{4}$$

$$\int_{\lambda_0}^{\lambda} \frac{d\lambda}{\sqrt{-2h\lambda^4 - 2\frac{fM}{c^3}\lambda^3 + 2(c_2 - h)\lambda^2 - 2\frac{fM}{c^3}\lambda + (2c_2 + c_1^2)}} = \tau + c_4, \tag{5}$$

$$w = c_1 \int_{w_0}^{w} \frac{(\lambda^2 + \mu^2)\,d\tau}{(1 - \mu^2)(1 - \lambda^2)} + c_5, \tag{6}$$

where h, c_1, c_2, c_3, c_4, c_5 are arbitrary integration constants, and λ_0, μ_0, w_0 are certain initial values of variables λ, μ, and w.

In the current paper we shall be concerned with the investigation of polar orbits, by which we understand the orbits lying in the Earth's meridian planes.

Equalities (2) show that the coordinate ω must be a constant quantity for polar orbits. Then it follows from the correlation (6) that $c_1 = 0$. Since the direction of the axis Ox is chosen arbitrarily, one may estimate without community limitation that $y = 0$, i.e. $c_5 = \pi/2$. That is why the position of the satellite will be determined by the coordinates x and z, which are connected with λ and μ by the formulae

$$x = c\sqrt{(1+\lambda^2)(1-\mu^2)}, \qquad z = -c\lambda\mu, \tag{7}$$

while λ and μ must be found from the quadratures

$$\int_{\mu_0}^{\mu} \frac{d\mu}{\sqrt{2h\mu^4 + 2(c_2 - h)\mu^2 - 2c_2}} = \tau + c_3, \tag{8}$$

$$\int_{\lambda_0}^{\lambda} \frac{d\lambda}{\sqrt{-2h\lambda^4 + 2(c_2 - h)\lambda^2 - 2\frac{fM}{c^3}\lambda^3 - 2\frac{fM}{c^3}\lambda + 2c_2}} = \tau + c_4. \tag{9}$$

Let us recall the geometrical meaning if the coordinates λ and μ. From the correlations (7) we find

$$\frac{x^2}{c^2(1+\lambda^2)} + \frac{z^2}{c^2\lambda^2} = 1, \tag{10}$$

$$\frac{x^2}{c^2(1-\mu^2)} - \frac{z^2}{c^2\mu^2} = 1. \tag{11}$$

The quality (10) shows that the equation $\lambda = \text{const}$ determines the family of ellipses whose center coincides with the center of the Earth,

and whose small semi-axis is perpendicular to the terrestrial equator plane. For $\mu = \text{const.}$, the equation (11) is the equation of hyperbola family.

1. QUALITATIVE INVESTIGATION OF POLAR ORBITS.

Formulae (8) and (9) may be rewritten in the form

$$\left(\frac{d\mu}{d\tau}\right)^2 = 2h(\mu^2 - 1)(\mu^2 - \delta^2), \tag{12}$$

$$\left(\frac{d\lambda}{d\tau}\right)^2 = -2h(\lambda^2 - 1)\varphi(\lambda), \tag{13}$$

where

$$\varphi(\lambda) = \lambda^2 + \frac{fM}{hc^3}\lambda - \frac{c_2}{h}; \qquad \delta^2 = -\frac{c_2}{h}.$$

Let us study first of all the regions of motion possibility. It is obvious that motion is only possible for three values of λ and μ for which the right-hand parts of the equalities (12) and (13) are non negative. When investigating the quadrature determining the quantity λ, we shall distinguish three cases:

1. The radicals of the polynomial $\varphi(\lambda)$ are real and different.
2. The radicals of $\varphi(\lambda)$ are equal.
3. The radicals of $\varphi(\lambda)$ are complex.

Let us designate these radicals by α and β. Then,

$$\alpha = -\frac{fM}{2hc^3} + \sqrt{\left(\frac{fM}{2hc^3}\right)^2 + \frac{c_2}{h}}, \tag{14}$$

$$\beta = -\frac{fM}{2hc^3} - \sqrt{\left(\frac{fM}{2hc^3}\right)^2 + \frac{c_2}{h}}. \tag{15}$$

Let us consider the first case. It may be seen from formula (13) that in the first case at $h > 0$ the motion is possible in the region determined by the inequality $\beta \leqslant \gamma \leqslant \alpha$, , and at $h < 0$ there are two regions of motion: $\lambda > \alpha$, $\lambda < \beta$.

As is shown by equalities (10), the geometrical dimensions of orbits increase as $|\lambda|$ increases. That is why at $h > 0$ we shall have motion in a bounded part of space, and at $h < 0$ the motion will take place in the unbounded part of the space.

Let now be $\alpha = \beta$. This is possible under the condition of fulfilment of the equality

$$-\frac{c_2}{h} = \left(\frac{fM}{2hc^3}\right)^2.$$

Thus

$$\delta = -\alpha = -\beta = \frac{fM}{2hc^3}.$$

Since for the satellite orbits $|\beta| > 1$, we shall also have $\delta > 1$. Consequently at $h < 0$ the right-hand part of the equality (12) will be nonnegative under the condition $|\mu| > 1$. But in a real motion $|\mu| < 1$. That is why if $h < 0$, the motion is impossible in the second case. If $h > 0$ the right-hand part of the equality (12) will be nonnegative under the condition that μ vary from -1 to $+1$. Formula (13) shows t at for $\alpha = \beta$, the motion is possible only at a constant

$$\lambda = -\frac{fM}{2hc^3}.$$

Let us now consider the case when the radicals of the polynomial $\varphi(\lambda)$ are complex. In this case the right-hand part of formula (13) will be nonnegative for all λ, if $h < 0$, and will be always negative if $h > 0$. Consequently, at $h \neq 0$ all the orbits are situated in the unbounded space.

Let be $h = 0$. Then

$$\left(\frac{d\lambda}{d\tau}\right)^2 = -(\lambda^2 + 1)\left(\frac{2fM}{c^3}\lambda - 2c_2\right).$$

whence it may be seen that $\left(\frac{d\lambda}{dt}\right)^2 > 0$, if $\lambda < \frac{c_2c^3}{fM}$. That is why even at $h = 0$ there are no motions taking place in the bounded part of space.

Therefore, motions in the bounded part of space may only take place for $h > 0$. If $h \leq 0$, all motions take place in the unbounded part of space.

The constant h is expressed through the initial conditions as follows [1]:

$$h = -\frac{1}{c^2}\left(\frac{1}{2}v_0^2 - U_0\right), \tag{16}$$

where v_0 and U_0 respectively are the satellite's velocity and the force function at the initial moment. Consequently the quantity $-\frac{1}{2}mhc^2$, where m is the mass of the satellite, is the total mechanical energy of the satellite; if it is negative, its motion takes place in the bounded part of space, if it is zero or positive, all orbits will drifts toward the infinity.

Let us rewrite the equality (16) in the form

$$h = -\frac{1}{c^2}\left\{\frac{1}{2}v_0^2 - \frac{fM}{r_0} - U_0 + \frac{fM}{r_0}\right\} \qquad (17)$$

and let us designate by mH the total energy in the unperturbed motion, i.e. we postulate

$$H = \frac{1}{2}v_0^2 - \frac{fM}{r_0}$$

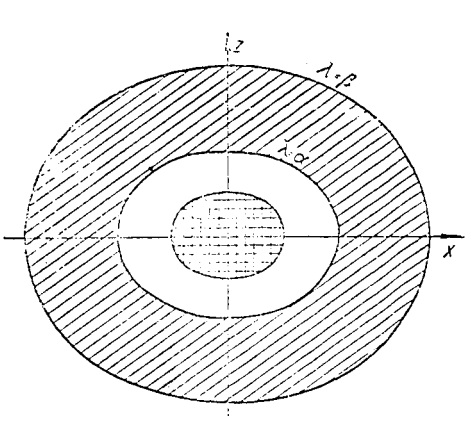

Fig. 1

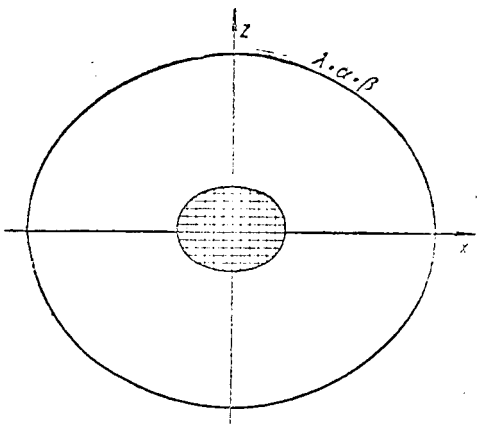

Fig. 2

Let us take for the initial moment the moment of time when the satellite crosses the Earth's axis of rotation. Then $x_0 = 0$, $z_0 = r_0$ and formula (17) will be written in the form

$$h = -\frac{1}{c^2}\left\{H + \frac{fM}{r_0}\frac{c^2}{r_0^2 + c^2}\right\}. \qquad (18)$$

Let be $H < 0$. The unperturbed motion of the satellite is then elliptical. Because the second component of formula (18) is positive, the constant h may be either positive or negative. Thus the perturbed motion may take place in the finite as well as in the infinite parts of space.

If $H \geqslant 0$, $h < 0$. That is why both, the unperturbed and the perturbed orbits will necessarily drift to infinity.

The preceding considerations may as a whole be formulated in the form of the following theorem:

THE KEPLER ELLIPTICAL MOTION MAY CONVERT INTO A MOTION TAKING PLACE IN INFINITE SPACE UNDER THE EFFECT OF PERTURBATIONS ONLY DEPENDENT ON EARTH'S NONSPEHRICITY. THE UNPERTURBED PARABOLIC AND HYPERBOLIC MOTIONS CANNOT CONVERT INTO LIMITED MOTION UNDER THE EFFECT OF THE SAME PERTURBATIONS.

In the current work we shall only be interested in orbits lying in the limited part of space. It follows from the preceding analysis that there are two cases of limited motions: the first, when the coordinate λ is included in certain bounded limits, and the second – when λ is a constant quantity. As was already shown, the coordinate μ varies in such case from -1 to $+1$.

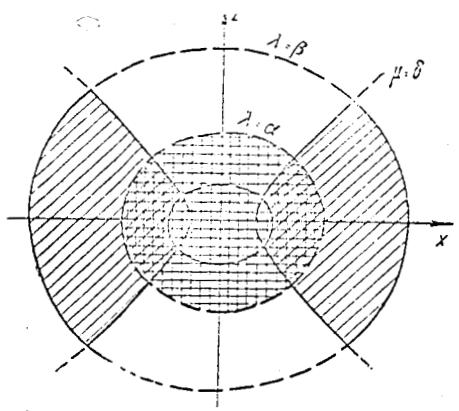

FIG 3.

Let us now pass to the investigation of the quadradture determining

the coordinate . Four cases must be considered here: a) $\delta > 1$; b) $\delta < 1$, $\delta \neq 0$; c) $\delta = 1$; d) $\delta = 0$. In the case a) the quantity may vary within the limits from -1 to $+1$. In the case b) μ varies within the limits $-\delta$ to $+\delta$. In the case c) μ varies from -1 to $+1$. Here there may a motion taking place with the constancy of μ. In the case d) motion also takes place with μ constancy.

If we join these four cases for μ with the already considered two cases for λ, we shall obtain the following five types of motion:

Type 1-a.- This type of motion takes place when the radicals of the polymomial $\varphi(\lambda)$ are real and different and when $\delta > 1$. The region of possible motion is determined by the inequalities $-1 \leqslant \mu \leqslant +1$. Consequently, in this case the satellite will always be situated in a space bounded by two ellipses.

Type 2-a.- This type of motion takes place when the radicals of the polynomial $\varphi(\lambda)$ are equal and when $\delta > 1$. This type includes the elliptical orbits.

In this case we have

$$\lambda = -\frac{fM}{2hc^3}, \qquad -1 \leqslant \mu \leqslant 1.$$

Type 1-b.- In this case $\alpha \neq \beta$ and $\delta < 1$. Thus $-\delta \leqslant \mu \leqslant \delta$, $\beta \leqslant \lambda \leqslant \alpha$, i.e. the region of possible motion is bounded by two ellipses $\lambda = \alpha$ and $\lambda = \beta$, and by two hyperbola branches $\mu = \delta$. The motion of

an artificial heavenly body will take place in such a fashion, that a collision with the Earth will necessarily occur. Since indeed $\delta < 1$,

$$|\alpha| = \left| -\frac{fM}{2hc^3} + \sqrt{\left(\frac{fM}{2hc^3}\right)^2 - \delta^2} \right| < 1,$$

and thus one of the ellipses lies inside the Earth.

Type 1-c.— We have in this case $-1 \leqslant \mu \leqslant +1$, $\beta \leqslant \lambda \leqslant \alpha$. Here too collision with the Earth will take place, for

$$|\alpha| = \left| -\frac{fM}{2hc^3} + \sqrt{\left(\frac{fM}{2hc^3}\right)^2 - 1} \right| \leqslant 1,$$

i.e. the trajectory will belong to the ballistic class.

Let us note that there exist such motions whenever the coordinate μ is constant. Indeed, $\mu = \pm 1$ is the solution of the equation (12) and of the equation which is obtained from (12) by way of its differentiation along τ. It follows then from formulae (7) that $x = 0$, $z = \pm c$. Consequently, the motion takes place in this case along the Earth's axis of rotation, with the distance from the center of the Earth oscillating within the limits $|\alpha c| \leqslant |z| \leqslant |c\beta|$.

Type 1 — d.- In this case $\mu = 0$, $\beta \leqslant \lambda \leqslant \alpha$. Thus $x = c\sqrt{1+\lambda^2}$, $z = 0$, i.e. the motion takes place along a cut of the straight line lying in the equator plane and passing through the center of the Earth. Since $\alpha = 0$, we have $c \leqslant x \leqslant c\sqrt{1+\beta^2}$.

The last three types of motion offer no interest in regard to

the theory of artificial satellite motion, for such motions lead necessarily to collision with the Earth. We shall designate this type of motions as ballistic motions. Those taking place in a bounded part of space and not leading to collision with the Earth, shall be designated as "satellite" motions.

The preceding analysis shows that two types of satellite orbits, included in an elliptic ring, exist in the meridian planes of the Earth: elliptical orbits and orbits included in an elliptical ring. We shall pause at these two types of orbits, examining them at further length.

2. ELLIPTICAL ORBITS.

The fulfilment of the following equalities is prerequisite for the existence of elliptical orbits :

$$\alpha = \beta = -\delta = -\frac{fM}{2hc^3}.$$

It follows from these equalities that the integration constants are not independent, but are linked by the following correlation:

$$c_2 = -h\left(\frac{fM}{2hc^3}\right)^2.$$

The elliptical coordinate λ is constant and is determined by the formula

$$\lambda = -\frac{fM}{2hc^3}.$$

Let us find the second coordinate. To do that let us rewrite the equality (12) in the form

$$\int_0^{\mu} \frac{d\mu}{\sqrt{(1-\mu^2)(1-k^2\mu^2)}} = m\tau, \tag{20}$$

where

$$k = \frac{1}{\delta}, \qquad m = \delta\sqrt{2h}.$$

If we inverse the elliptical quadrature (20), we shall obtain

$$\mu = \mathrm{sn}(m\tau, k). \tag{21}$$

Substituting in formulae (7) and by their values from (19) and (21), we shall obtain the following expressions for the rectangular coordinates of the satellite:

$$x = a\,\mathrm{cn}\,m\tau, \qquad z = b\,\mathrm{sn}\,m\tau, \tag{22}$$

where

$$a = c\sqrt{1+\frac{fM}{2hc^3}}, \qquad b = c\,\frac{fM}{2hc^3}.$$

Obviously, the constants a and b are respectively the great and the small semi-axes of the elliptical orbit. The relationship between the orbit's excentricity and its great semi-axis is given by the formula

$$e = \frac{c}{a}.$$

The module of the elliptical functions k is expressed through the eccentricity as follows:

$$k = \frac{e}{\sqrt{1-e^2}}.$$

Since the quantity is small by comparison with the great semi-axis of the orbit, it follows that both the eccentricity and the module are small quantities.

Let us find the formula linking the time t with the regulating variable τ. First of all we have

$$t - t_0 = \lambda^2 \tau + \int_0^{\tau} \mu^2 d\tau,$$

where t_0 is a certain initial moment.

Substituting formula (21) into the equality (23) and assuming

$$s = \operatorname{sn} m\tau ,$$

we then obtain

$$t - t_0 = \lambda^2 \tau + \frac{1}{m} \int_0^s \frac{s^2\, ds}{\sqrt{(1 - s^2)(1 - k^2 s^2)}} .$$

The last equality may be rewritten in the form

$$t - t_0 = \lambda^2 \tau + \frac{1}{mk^2} \{F(\operatorname{am} m\tau, k) - E(\operatorname{am} m\tau, k) \qquad (24)$$

where F and E are the ellitpical integrals of the first and second kind. Formula (24) thus gives us the dependence between t and τ, searched for.

It follows from formuale (22) that the motion along the elliptical orbit is periodical relative to τ with a period equal to $4K/m$, where K is the total elliptical integral of the first kind.

To this interval of variable τ variation corresponds the following time interval:

$$T = \frac{4k^2}{m} K + \frac{1}{mk^2}(K - E),$$

where E is the total elliptical integral of the second kind.

The quantities K and E may be expanded in power series of the module k and consequently also in powers of orbit's eccentricity. After effecting the necessary operations we shall obtain

$$T = \frac{2\pi}{n}\left\{1 + \frac{9}{8}e^2 + \frac{3}{2}e^4 + \dots\right\}, \qquad (25)$$

where

$$n = \sqrt{\frac{fM}{a^3}}.$$

Consequently, the rotation period of the satellite around the Earth depends not only on the mass of the Earth, but also on its ablation.

3. MOTION INSIDE THE ELLIPTICAL RING

The following type of motion was shown in section 2: The satellite moves all the time inside a region bounded by two ellipses. The coordinates λ and μ are in this case limited, and satisfying the following conditions:

It is easy to see that the great semi-axis and the eccentricity of the inner ellipse are expressed through the radical α as follows:

$$a_1 = c\sqrt{1+\alpha^2}, \qquad e_1 = \frac{1}{\sqrt{1+\alpha^2}}. \qquad (26)$$

The outer ellipse is determined by the following parameters:

$$a_2 = c\sqrt{1+\beta^2}, \qquad e_2 = \frac{1}{\sqrt{1+\beta^2}}. \tag{27}$$

In this case the coordinate μ varies as in the case of elliptical motion, i.e.

$$\mu = \operatorname{sn}(m\tau, k), \tag{28}$$

where

$$m = \delta\sqrt{2h}, \qquad k = \frac{1}{\delta}. \tag{29}$$

Making use of the Tables by Ryzhik and Gradshteyn [2] we find

$$\lambda = \frac{A + B\operatorname{cn}[\omega(\tau-\tau_0), \bar{k}]}{C + D\operatorname{cn}[\omega(\tau-\tau_0), \bar{k}]}, \tag{30}$$

where

$$A = \beta\sqrt{1+\alpha^2} + \alpha\sqrt{1+\beta^2}, \quad C = \sqrt{1+\beta^2} + \sqrt{1+\alpha^2},$$
$$B = \beta\sqrt{1+\alpha^2} - \alpha\sqrt{1+\beta^2}, \quad D = \sqrt{1+\alpha^2} - \sqrt{1+\beta^2},$$
$$k^2 = \frac{1}{2}\left\{1 - \frac{1+\alpha\beta}{\sqrt{(1+\alpha^2)(1+\beta^2)}}\right\},$$
$$\omega = \sqrt[4]{4h^2(1+\alpha^2)(1+\beta^2)}.$$

Substituting formulae (28) and (30) into the equality (7) and expressing the constants A, B, C and D through the parameters of the ellipses, we find after certain simplifications:

$$x = 2N\frac{\operatorname{dn}[\omega(\tau-\tau_0), \bar{k}]}{1-\gamma\operatorname{cn}[\omega(\tau-\tau_0), \bar{k}]}\operatorname{cn}(m\tau, k),$$

$$z = N\frac{\sqrt{1-e_1^2} + \sqrt{1-e_2^2} - (\sqrt{1-e_1^2} - \sqrt{1-e_2^2})\operatorname{cn}[\omega(\tau-\tau_0), \bar{k}]}{1-\gamma\operatorname{cn}[\omega(\tau-\tau_0), \bar{k}]}\operatorname{sn}(m\tau, k)$$

where

$$N = -\frac{a_1a_2}{a_1+a_2}, \quad \gamma = \frac{a_2-a_1}{a_2+a_1}.$$

For the constants m, ω, k and $\bar{k}$ we have the following expressions:

$$m = \sqrt{\frac{fM}{c^4}\,\frac{a_1\sqrt{1-e_1^2}+a_2\sqrt{1-e^2}}{2}},$$

$$\omega = \sqrt{\frac{fM}{c^4}\,\frac{2a_1a_2}{a_1\sqrt{1-e_1^2}+a_2\sqrt{1-e_2^2}}},$$

$$k^2 = \frac{4e_1^2e_2^2}{(e_1\sqrt{1-e_1^2}+e_2\sqrt{1-e_2^2})^2},$$

$$\bar{k}^2 = \frac{1}{2}\left(1+e_1e_2-\sqrt{(1-e_1^2)(1-e_2^2)}\right).$$

The last equalities show that even in this case the modules of elluptical integrals are small quantities. Consequently, convenient trigonometrical series may be obtained for the coordinates of the satellite. We must expand to that effect the elliptical functions in series by sines and cosines, multiples of $\tau-\tau_0$, where τ_0 is a certain constant. Because of the smallness of k and $\bar{k}$, the obtained series will converge rapidly. We propose to undertake such an expansion for the general case of satellite motion in a subsequent work.

**** THE END ****

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CHAIR OF
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